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MTH 213, Exam II

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$$Score = \frac{45}{45}$$ Excellent

QUESTION 1. (10 points)
Out of 18 available persons (ten males and eight females), a committee with seven persons is formed. Assume that $f_1, f_2, ..., f_8$ are the names of the females and $m_1, m_2, ..., m_{10}$ are the names of the males.

(i) If f_4, f_7, m_1 and m_7 must be in the committee, in how many different ways can we form such a committee?

$= 1C1 \cdot 1C1 \cdot 1C1 \cdot 1C1 \cdot 14C3 = \binom{14}{3} = \cancel{364}$ no need

(ii) If f_3, f_5, and exactly 2 males must be in the committee, in how many different ways can we form such a committee?

$= (1C1)\cdot(1C1)\cdot(10C2)\cdot(6C3) = \binom{10}{2}\binom{6}{3} = \cancel{45 \times 20} = \cancel{900}$ no need

(iii) If m_2 or f_7, but not both, must be in the committee, in how many different ways can we form such a committee?

$(1C1)(16C_6) + 1C1 \cdot (^{16}C_6) = 2(16C6) = 2\binom{16}{6} = 16016$

(iv) If the committee must have m_3, m_9 and exactly 2 females, in how many different ways can we form such a committee?

$(1C1)(1C1)(^8C2)(8C3) = \binom{8}{2}\binom{8}{3} = 28 \times 56 = 1568$ no need

(v) If the committee must have m_3, m_7 and at least 2 females, in how many different ways can we form such a committee? $m_3m_7 2W\ 3M + m_3m_7 3W\ 2M + m_3m_7 4W\ 1M + m_3m_7 5W$

$= (8C2)(8C3) + (8C3)(8C2) + (8C4)(8C1) + (8C5)$
$= 3752$

QUESTION 2. (6 points) The digits 0, 1, 2, 3, . . . , 9 are used to construct 5-digits car plates (note that 10 digits are available).

(i) If a digit cannot be repeated in a plate number, the first digit must be odd, and the fifth digit must be even; how many car plates can be constructed?

Odd Even
5 8 7 6 5 $= 8 \times 7 \times 6 \times 5 \times 5 = 8400$

(ii) If repetition is allowed and at least one digit must be 0; how many car plates can be constructed? (note that 00000, 00101, 03060, 06600,66640, and so on is allowed) 1 zero + 2 zero + 3 zero + 4 zero + 5 zero

1 1

$= 5 \times 9^4 + (5C2)(9^3) + (5C3)9^2 + (5C4)9 + 1$
$= 40951$ Good

(iii) If repetition is allowed, exactly 2 digits are 1, and exactly one digit is 5; how many car plates can be constructed? (note that 11075, 50101, 22511, 01815, and so on is allowed)

two 1 + one 5

1 1 5

$= (5C2)(3C1)(8^2)$ Excellent!

$= 1920$

QUESTION 3. (6 points) Let $a_n = 7a_{n-1} - 12a_{n-2} + 7(5^n)$. Find a general formula for a_n, no need to find c_1, c_2.

$a_n - 7a_{n-1} + 12a_{n-2} = 7(5^n)$

(1) To find $a_h(n)$: homogeneous part. Let $a_n - 7a_{n-1} + 12a_{n-2} = 0$

$[\alpha^n - 7\alpha^{n-1} + 12\alpha^{n-2} = 0] \div \alpha^{n-2}$

$\alpha^2 - 7\alpha + 12 = 0$

$\alpha^2 - 3\alpha - 4\alpha + 12 = 0$

$\alpha(\alpha - 3) - 4(\alpha - 3) = 0$

$(\alpha - 3)(\alpha - 4) = 0$

$\alpha = 3, 4$

$a_h(n) = c_1(3)^n + c_2(4)^n$

(2) $a_p(n) = A(5^n)$

$a_p(n) - 7a_p(n-1) + 12a_p(n-2) = 7(5^n)$

$A5^n - 7A \cdot 5^{n-1} + 12A5^{n-2} = 7(5^n)$

$5^n\left(A - \frac{7}{5}A + \frac{12}{25}\right) = 7(5^n)$

$\frac{2}{25}A = 7 \longrightarrow A = \frac{175}{2} \longrightarrow a_p(n) = \frac{175}{2}(5^n)$

$a_n = a_p(n) + a_h(n)$

$$\boxed{a_n = c_1(3^n) + c_2(4^n) + \frac{175}{2}(5^n)}$$

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QUESTION 4. (6 points) Let $a_n = -6a_{n-1} + 7a_{n-2} + 5n$. Find a general formula for a_n, no need to find c_1, c_2,

$a_n + 6a_{n-1} - 7a_{n-2} = 5n$

(1) To find $a_h(n)$: $a_n + 6a_{n-1} - 7a_{n-2} = 0$

$[\alpha^n + 6\alpha^{n-1} - 7\alpha^{n-2} = 0] \div \alpha^{n-2}$

$\alpha^2 + 6\alpha - 7 = 0$

$\alpha^2 + 7\alpha - \alpha - 7 = 0$

$\alpha(\alpha + 7) - (\alpha + 7) = 0$

$(\alpha - 1)(\alpha + 7) = 0$

$\alpha = 1, -7$

$a_h(n) = c_1(1)^n + c_2(-7)^n$

(2) $a_p(n) = [bn]n = bn^2$ (as one $\alpha = 1$ & its polynomial)

$a_p(n) + 6a_p(n-1) - 7a_p(n-2) = 5n$

$bn^2 + 6b(n-1)^2 - 7b(n-2)^2 = 5n$

$bn^2 + 6[bn^2 - 2bn + b] - 7[bn^2 - 4bn + 4b] = 5n$

$-12bn + 28bn + 6b - 28b = 5n$

continuation

$\rightarrow 16b = 5$

$b = \frac{5}{16}$

$a_n = a_h(n) + a_p(n)$

$a_n = c_1(1)^n + c_2(-7)^n + \frac{5}{16}n^2$

▶ P.T.O

$a_p(n) = [bn + c]n = bn^2 + cn$ (as $\alpha = 1$ & it's a poly.)

$bn^2 + cn + 6[b(n-1)^2 + c(n-1)] - 7[b(n-2)^2 + c(n-2)] = 5n$

$\cancel{bn^2} + cn + 6[\cancel{bn^2} - 2bn + b + cn - c] - 7[\cancel{bn^2} - 4bn + 4b + cn - 2c]$

$(\cancel{cn} + 6cn - 7cn) - 12bn + 28bn + 6b - 28b - 6c + 14c = 5n$

$16bn - 22b + 8c = 5n$

$16b = 5 \qquad -22b + 8c = 0$

$\boxed{b = \frac{5}{16}} \qquad 22b = 8c$

$c = \frac{22}{8} \times \frac{5}{16} = \boxed{\frac{55}{64}}$

$\boxed{a_p(n) = \frac{5}{16}n + \frac{55}{64}}$

$\boxed{a_n = c_1(1)^n + c_2(-7)^n + \frac{5}{16}n^2 + \frac{55}{64}n}$

QUESTION 5. (6 points)

(i) What is the minimum number of chocolate bags that can be distributed over 24 schools so that a school will have at least 12 bags of chocolate?

$\left\lceil \frac{n}{24} \right\rceil = 12$ $n = 24 \times 11 + 1$
$= 265$

(ii) There are 876 positive integers such that each integer is of form 5k for some integer $k \in Z$. Then there are at least k integers out of the 876 numbers, say, $n_1, \ldots, n_k$ such that $n_1 \pmod{20} = n_2 \pmod{20} = \cdots = n_k \pmod{20}$. What is the best value of k?

m

Domain : 876 +ve integers

Co-domain : $\{0, 5, 10, 15\}$

Best value (m) = $\left\lceil \frac{876}{4} \right\rceil = 219$

QUESTION 6. (4 points) Consider the following permutation function

$$f: \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 1 & 4 & 5 & 3 & 7 & 8 & 9 & 10 & 6 \end{pmatrix}$$

Find the least positive integer m such that $f^m = f \circ f \circ \cdots \circ f = I$, the identity function.

Disjoint cycles = $\underbrace{(1\ 2)}_{2\text{ cycles}}\underbrace{(3\ 4\ 5)}_{3\text{ cycle}}\underbrace{(6\ 7\ 8\ 9\ 10)}_{5\text{ cycle}}$

$m = LCM(2, 3, 5)$ $LCM(2,3) = 6$
$= LCM(6, 5)$
$= 30$

QUESTION 7. (7 points) Let $A = \{2, \{2\}, 5, 8, 11\}$ and $B = \{4, 7, 2, 11, \{2\}\}$. Then

(a) Find $A - B$ = $\{5, 8\}$

(b) Write down T or F

(i) $\{2, \{2\}, 11\} \in \mathbb{P}(A)$. True as $\{2, \{2\}, 11\} \subseteq A \to T$

(ii) $\{2\} \in B$ True

(iii) $\{8, \{2\}, 11\} \subset A$ True

(iv) $\{\{2\}\} \subset \mathbb{P}(B)$ True $\{2\} \in P(B) \to \{2\} \subset A \to T$

(v) $\{\{7,4\}, \{4\}\} \subset \mathbb{P}(B)$ True $\{7,4\}, \{4\} \in P(B) \to \subset B \to T$

(vi) $(7, 11) \in A \times B$ False $7 \notin A$